MAGNETOHYDRODYNAMIC METHODS OF MEASURING MASS VELOCITY AND ELECTRICAL CONDUCTIVITY PARAMETERS VARYING ALONG THE DIRECTION OF THE FLOW

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Contact measurements of varying mass velocity and electrical conductivity in MHD channels are considered. The conditions under which the mass-velocity profile may be measured are found, and the measuring error is estimated. It is shown that MHD measurements of electrical conductivity may be made even when the mass-velocity profile is not known in advance.

1. During the motion of a conducting medium in an external magnetic field an electric field depending on the velocity of the motion is created. If this velocity is constant, it may be determined by measuring the potential difference between two electrodes placed in the flow. When a resistance is connected between the electrodes, the current arising from this potential difference depends on the resistance of the medium between the electrodes and hence on the electrical conductivity of the medium; by measuring the current we may find the electrical conductivity.

A theory presented in two earlier expositions [1, 2] related to measurements of velocity in MHD channels, in which allowance was made for the nonuniformity of the magnetic field and the changes taking place in the velocity and electrical conductivity on passing across the channel. Considerable changes may often take place in these quantities on passing along the channel as well. For example, when a detonation wave passes along such a channel, measurements of the electrical-conductivity and mass-velocity profiles might provide useful information regarding the chemical reactions in the explosive. Changes taking place in the velocity along the direction of motion have been recorded in experiments with shock tubes [3] and high-speed gas flows [4]. However, the possibility that changes in velocity might influence the results of the measurements has never been considered in existing experimental investigations.

A solution was given in [2] to the problem of the electric field in a channel of constant electrical conductivity, due to allowance being made for the changes taking place in the velocity on passing along the channel. It would be a very complicated matter to solve the coresponding problem if the electrical conductivity also depended in an arbitrary manner on the longitudinal coordinate.

In order to determine the possibility of measuring the electrical-conductivity and velocity profiles magnetohydrodynamically it is unnecessary to obtain an exact solution of this problem. In this paper we shall find the necessary conditions for making such measurements and shall estimate the errors associated with changes in velocity along the channel.

2. We shall consider that the measurements are carried out in an MHD channel of rectangular cross section as indicated in Fig. 1, although all the results will be applicable to channels of arbitrary cross section. The conducting medium moves along the channel, its velocity at a particular instant of time being $v = (v(x), 0, 0)$. The uniform external magnetic field H is directed along the z axis (perpendicular to the plane of the figure). The electrical conductivity of the medium is $\sigma = \sigma(x)$. The electrode 1 is grounded; electrode 2 is connected to it through a resistance R. The walls of the channel are insulators. The quantity

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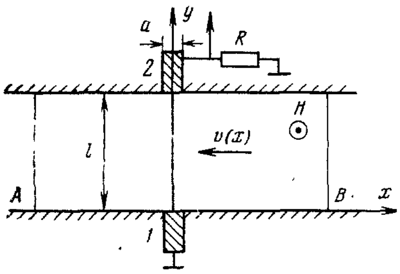

Fig. 1

to be measured is the voltage V between the electrodes. It is required to determine the dependence of the voltage on the velocity and electrical-conductivity distribution in the flow.

Subsequently we shall make the following assumptions:

1) The electrical conductivity of the medium $\sigma \gg \tau^{-1}$, where τ is the characteristic time of the changes taking place in the profiles v(x) and $\sigma(x)$, i.e., the time required for these quantities to change in a certain cross section of the channel (steady-state problem);

2) the effect of the magnetic field on the motion of the medium may be neglected, and so may the induced magnetic field, so that the velocity of the medium and the magnetic field may be regarded as prespecified;

3) Ohm's law for the medium may be written in the form

$$\mathbf{j} = \sigma(\mathbf{E} + (1/c)[\mathbf{v}, \mathbf{H}])$$

Conditions 1-3 are satisfied for a wide class of phenomena.

It was shown in [2] that on satisfying conditions 1-3 the electric field $\mathbf{E}$ in the medium was of the potential type, $\mathbf{E} = -\nabla\varphi$, while the potential φ satisfied the equation

$$\Delta\varphi + \frac{1}{\sigma}(\nabla\sigma\nabla\varphi) = \frac{1}{c}\left(\frac{\nabla\sigma}{\sigma}[\mathbf{v}, \mathbf{H}]\right) + \frac{1}{c}(\mathbf{H} \operatorname{rot} \mathbf{v})$$

In the present geometry this equation assumes the form

$$\Delta\varphi + \frac{1}{\sigma}\frac{d\sigma}{dx}\frac{\partial\varphi}{\partial x} = 0 \tag{2.1}$$

The boundary conditions for φ on the insulating walls are

$$(\nabla\varphi)_n = (1/c)([\mathbf{v}, \mathbf{H}])_n$$

The potential of electrode 1 $\varphi(1) = 0$, that of electrode 2 $\varphi(2) = V$. We also assume that in channel cross sections A and B the x component of the current density equals zero, i.e., $\partial\varphi/\partial x = 0$ (for example, at the boundaries of a conducting gas lock in a shock tube); these cross sections may occur at $x = \pm\infty$.

After solving (2.1) subject to the boundary conditions indicated, we may determine the potential V of electrode 2 from Ohm's law for the external circuit

$$\int\limits_{S(2)} \sigma\left(-\frac{\partial\varphi}{\partial y} + \frac{vH}{c}\right) ds = \frac{V}{R} \tag{2.2}$$

where S(2) is the surface of electrode 2.

Let us denote the characteristic value of the velocity in the flow by v_0 and seek φ in the form

$$\varphi = \frac{Hyv}{c} + \left(V - \frac{Hlv_0}{c}\right)\psi_0 + \frac{Hlv_0}{c}\psi_1 \tag{2.3}$$

Here l is the distance between electrodes 1 and 2 (Fig. 1) and ψ_0 and ψ_1 are the solutions of the following boundary problems:

$$\Delta\psi_0 + \frac{1}{\sigma}\frac{d\sigma}{dx}\frac{\partial\psi_0}{\partial x} = 0, \quad \psi_0(2) = 1 \tag{2.4}$$

$$\Delta\psi_1 + \frac{1}{\sigma}\frac{d\sigma}{dx}\frac{\partial\psi_1}{\partial x} = -\frac{y}{lv_0}\left(\frac{d^2v}{dx^2} + \frac{1}{\sigma}\frac{d\sigma}{dx}\frac{dv}{dx}\right) \tag{2.5}$$

$$\psi_1(2) = 1 - \frac{v}{v_0}, \quad \frac{\partial\psi_1}{\partial x} = -\frac{y}{lv_0}\frac{dv}{dx} \text{ on } A, B$$

(Here and subsequently, if the boundary condition is not written out, it should be regarded as zero.)

It is easy to see that, if ψ_0 and ψ_1 are solutions of the problems (2.4) and (2.5), then the potential φ determined from (2.3) will satisfy (2.1) and the boundary conditions.

From Eq. (2) defining V we obtain

$$-\left(V - \frac{Hlv_0}{c}\right)\int\limits_{S(2)} \sigma\frac{\partial\psi_0}{\partial y} ds - \frac{Hlv_0}{c}\int\limits_{S(2)} \sigma\frac{\partial\psi_1}{\partial y} dS = \frac{V}{R}$$

or

$$
\begin{gathered}
-\left(V-\frac{Hlv_0}{c}\right)\frac{1}{R_\sigma}-\frac{Hlv_0}{c}\alpha=\frac{V}{R} \\
R_\sigma=\left(\int\limits_{S(2)}\sigma\frac{\partial\psi_0}{\partial y}ds\right)^{-1}, \qquad \alpha=\int\limits_{S(2)}\sigma\frac{\partial\psi_1}{\partial y}ds
\end{gathered} \tag{2.6}
$$

The quantity R_σ is the resistance of the conducting medium between electrodes 1 and 2. This follows from the boundary problem (2.4): R_σ^{-1} is the current to the electrode 2 when the voltage on the latter is unity. From (2.6) we obtain

$$
V=\frac{Hlv_0}{c}\frac{R}{R+R_\sigma}(1-\alpha R_\sigma) \tag{2.7}
$$

For a constant velocity of the medium, $v(x) = v_0$, the righthand sides of the equation and boundary conditions (2.5) vanish, so that $\psi_1 = 0$ and $\alpha = 0$. Equation (2.7) than transforms into $V = Hlv_0c^{-1}R/(R + R_\sigma)$.

In Eq. (2.7) V and R_σ are determined experimentally; we cannot determine α since α depends on the profiles $v(x)$ and $\sigma(x)$, which are not known in advance.

3. According to Eq. (2.7) the measured voltage V depends on the characteristic rate of flow v_0. This may seem strange at first sight since the choice of v_0 is arbitrary. After considering the boundary problems (2.4) and (2.5) we easily see that the product $v_0(1-\alpha R_\sigma)$ does not depend on the choice of v_0. At a specified instant of time the characteristic velocity of the flow may thus be regarded as the velocity for $x = 0$, i.e., that in the middle of the electrodes. After measuring V we may use (2.7) to find the velocity v_0 in a specified ($x = 0$) cross section of the channel at a specified instant of time, subject to the condition $\alpha R_\sigma \ll 1$. The error in determining the velocity due to its variations along the channel is $\delta v/v = \alpha R_\sigma$.

The error will be small for a fairly slow change in velocity along the channel. We shall shortly show that measurements may validly be made if the velocity varies little over distances of the order of the width of the channel l and the width of the electrode a (Fig. 1), i.e., the conditions $l \ll \Delta$, $a \ll \Delta$, where Δ is the characteristic dimension of the change in velocity.

The quantity α is determined by the righthand side of the equation and by the boundary conditions of the boundary problem (2.5) for the potential ψ_1. Let us express ψ_1 in the form $\psi_1 = \psi_{11} + \psi_{12}$, where ψ_{11} is the solution to the problem

$$
\Delta\psi_{11}+\frac{1}{\sigma}\frac{d\sigma}{dx}\frac{\partial\psi_{11}}{\partial x}=0, \qquad \psi_{11}(2)=1-\frac{v}{v_0} \tag{3.1}
$$

and ψ_{12} is the solution to the problem

$$
\begin{gathered}
\Delta\psi_{12}+\frac{1}{\sigma}\frac{d\sigma}{dx}\frac{\partial\psi_{12}}{\partial x}=-\frac{y}{lv_0}\left(\frac{d^2v}{dx^2}+\frac{1}{\sigma}\frac{d\sigma}{dx}\frac{dv}{dx}\right) \\
\frac{\partial\psi_{12}}{\partial y}=-\frac{y}{lv_0}\frac{dv}{dx}\ \text{on}\ A,\ B
\end{gathered} \tag{3.2}
$$

Then

$$
\alpha R_\sigma=R_\sigma\int\limits_{S(2)}\sigma\frac{\partial\psi_{11}}{\partial y}ds+R_\sigma\int\limits_{S(2)}\sigma\frac{\partial\psi_{12}}{\partial y}ds=\alpha_1R_\sigma+\alpha_2R_\sigma
$$

We estimate the products $\alpha_1 R_\sigma$ and $\alpha_2 R_\sigma$ individually. Since

$$
\psi_{11}(2)=1-\frac{v}{v_0}\sim\frac{x}{v_0}\frac{dv}{dx}\sim\frac{a}{\Delta}
$$

while $\psi_0(2) = 1$, we find that α_1R_σ is of the order of a/Δ. Let us estimate α_2R_σ. We shall assume that the velocity $v(x)$ changes little over a distance of the order of l, i.e., $l \ll \Delta$, and that the characteristic dimension for the change in electrical conductivity is equal to (or exceeds) Δ in order of magnitude. The righthand side of (3.2) is then of the order of Δ^{-2}. At distances of the order of l from the electrodes $\Delta\psi_{12} \sim \psi_{12}/l^2$, since l is the characteristic dimension along y, and hence $\psi_{12} \sim l^2/\Delta^2$, while $\partial\psi_{12}/\partial y \sim l/\Delta^2$. The derivative $\partial\psi_0/\partial y \sim 1/l$.

Let us consider the region of flow $G(l \geq y \geq l/2,\ |z| \leq b/2,\ B \geq x \geq A)$, where b is the width of the channel in the direction z, $b \leq l$. If we multiply Eq. (3.2) by σ and integrate over the volume of the region G, using the Gauss theorem and the boundary conditions on planes A and B and the insulating walls, we obtain

$$\int\limits_{S(2)} \sigma \frac{\partial \psi_{12}}{\partial y} ds = \int\limits_{D} \sigma \frac{\partial \psi_{12}}{\partial y} ds$$

where the first integral is taken over the surface of the electrode 2 and the second over the boundary $D(y = l/2)$ of the region G. In the same way we may obtain

$$\int\limits_{S(2)} \sigma \frac{\partial \psi_0}{\partial y} ds = \int\limits_{D} \sigma \frac{\partial \psi_0}{\partial y} ds \tag{3.3}$$

Hence

$$\alpha_2 R_\sigma = \left(\int\limits_{D} \sigma \frac{\partial \psi_{12}}{\partial y} ds\right)\left(\int\limits_{D} \sigma \frac{\partial \psi_0}{\partial y} ds\right)^{-1} \tag{3.3}$$

The transition to integration over the median plane of flow D is convenient because on this $\partial\psi_{12}/\partial y$ and $\partial\psi_0/\partial y$ have no singularities.

The derivatives $\partial\psi_{12}/\partial y$ and $\partial\psi_0/\partial y$ behave differently when $|x| \gg l$: $\partial\psi_{12}/\partial y$ is of the order of l/Δ^2 in the region in which $v^{-1}dv/dx \sim \Delta^{-1}$, i.e., when $x \lessapprox \Delta$, while $\partial\psi_0/\partial y$ falls as x^{-2} when $|x| \gg l$, so that the main contribution to the integral in the denominator of (3.3) comes from that part of the region D in which $|x|\, a \leq l$ (we assume that $l \sim l$).

These considerations enable us to estimate the error

$$\alpha_2 R_\sigma \sim \sigma \frac{\partial \psi_{12}}{\partial y} b\Delta \left(\sigma \frac{\partial \psi_0}{\partial y} bl\right)^{-1} \sim \frac{l}{\Delta^2} \Delta \left(\frac{1}{l} l\right)^{-1} \sim \frac{l}{\Delta}$$

For the total error committed in MHD measurements of a velocity varying along the direction of flow we obtain

$$\delta v / v = (\alpha_1 + \alpha_2) R_\sigma = O(a / \Delta) + O(l / \Delta) \tag{3.4}$$

It follows from this that it is appropriate to use thin electrodes ($a \ll l$). Subject to this condition

$$\delta v / v = O(l / \Delta) \tag{3.5}$$

Magnetohydrodynamical measurements of mass velocity should therefore be made subject to the condition $a \ll l \ll \Delta$; then the order of magnitude of the error will be given by Eq. (3.5). If $l \ll \Delta$ and $a \sim l$, the error will be determined by Eq. (3.4). For example, in [4] the situation was $a/\Delta \approx 10^{-2}$, $l/\Delta \sim 10^{-1}$, and the error $\sim 10\%$.

When $l \gg \Delta$ we may use a similar argument to obtain the relation $\delta v/v = O(1)$ instead of (3.4). Thus for measuring the mass-velocity profile in, for example, a detonation wave the MHD method is unsuitable, since the detonation wave may only propagate through the charge when $l \gg \Delta$.

4. From Eq. (2.7) we may determine the current I following through the conducting medium

$$I = \frac{V}{R} = \frac{Hlv_0}{c} \frac{1 - \alpha R_\sigma}{R + R_\sigma} \tag{4.1}$$

Let us consider the quantity

$$U = \frac{Hlv_0}{c} - V = \frac{Hlv_0}{c} \frac{R_\sigma (1 + \alpha R)}{R + R_\sigma} \tag{4.2}$$

For $v(x) = v_0$, U equals the fall in voltage in the resistance of the conducting medium R_σ. Eliminating the parameters R from (4.1) and (4.2), we obtain a relationship for U(I)

$$U = R_\sigma I + Hlv_0 c^{-1} \alpha R_\sigma \tag{4.3}$$

i.e., a straight line, which, for $\alpha \neq 0$ (variable velocity), does not pass through the origin of coordinates.

Straight lines of this type were obtained in [5]. (The authors of [5] considered that the mass velocity in the reaction zone of a detonating expolsive was constant, disagreeing with the results of [6], and explained the "threshold voltage" U(O) as being due to effects close to the electrode in the presence of a magnetic field.) It follows from the foregoing that the slope of the straight Lines U(I) obtained in [5] does in fact represent the resistance between the electrodes, while the threshold voltage $U(0) = Hlv_0c^{-1}\alpha R_\sigma$, i.e., the deviation of these straight lines from the volt–ampere characteristic $U = R_\sigma I$, is due to the changes in velocity v(x) along the reaction zone. We note that, although the geometry of the epxeriments described in [5] differs from that indicated in Fig. 1, all the discussions of section 2, and in particular Eq. (2.7), remain valid; the estimate of U(o) agrees with the data of [5].

Hence, a measurement of the resistance of a conducting medium between two electrodes is perfectly possible for a variable flow velocity v; it is determined from the slope of the experimental U(I) curves, and a knowledge of the v(x) profile is not needed. Using the known time dependence of the resistance R_σ during the motion of the medium past the electrodes, we may find the electrical-conductivity profile. This part of the problem will not be considered here. One version of the solution appears in [7].

We note that the MHD method of measuring electrical conductivity has no advantages over the electrical-contact method [8], and is far more complicated owing to the necessity of creating a magnetic field; its use is thus only justified in rare cases, for example, when verifying the results of other methods.

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